Lecture given at the 61st Meeting of the

European Working Group

“Multicriteria Aid for Decisions"

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Ghent University, Belgium

Cycle-transitivity and the Resolution of Preference Cycles

Bernard DE BAETS and Hans DE MEYER

partially based on joint work with B. De Schuymen
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1. Statistical preference and cycles

Integers 1–18 distributed over 3 dice:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
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<tbody>
<tr>
<td></td>
<td>1</td>
<td>3</td>
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<td>15</td>
<td>16</td>
<td>17</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
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<td>C</td>
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1. Statistical preference and cycles

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B. De Baets
1. Statistical preference and cycles

Statistical preference:

- **$X \triangleright Y$**: $X$ is preferred to $Y$ if $\text{Prob}\{X > Y\} > \frac{1}{2}$

- can contain cycles (of type SCISSORS–STONE–PAPER)
1. Statistical preference and cycles

- **Statistical preference:**
  - $X \triangleright Y$: $X$ is preferred to $Y$ if $\text{Prob}\{X > Y\} > \frac{1}{2}$
  - can contain cycles (of type SCISSORS–STONE–PAPER)

![Diagram showing cycles and probability arrows]
1. Research questions

Question 1:
Does there exist a general framework for studying transitivity?
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Question 2:
Are cycles really incompatible with any notion of transitivity?
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Question 1:
Does there exist a general framework for studying transitivity?

Question 2:
Are cycles really incompatible with any notion of transitivity?

Question 3:
Can cycles be resolved?
2.1 Reciprocal relations

Reciprocal relation \( Q \) on \( X: Q : X^2 \rightarrow [0, 1] \) such that

\[
Q(a, b) + Q(b, a) = 1
\]
2.1 Reciprocal relations

Reciprocal relation $Q$ on $X$: $Q : X^2 \rightarrow [0, 1]$ such that

$$Q(a, b) + Q(b, a) = 1$$

Synonyms: probabilistic relation, ipsodual relation
2.1 Reciprocal relations

- Reciprocal relation $Q$ on $X$: $Q : X^2 \rightarrow [0, 1]$ such that
  \[ Q(a, b) + Q(b, a) = 1 \]

- Synonyms: probabilistic relation, ipsodual relation

- Generalization of complete relations:
  \[
  Q(a, b) = \begin{cases} 
  1 & \text{, if } R(a, b) = 1 \text{ and } R(b, a) = 0 \\
  1/2 & \text{, if } R(a, b) = R(b, a) = 1 \\
  0 & \text{, if } R(a, b) = 0 \text{ and } R(b, a) = 1
  \end{cases}
  \]
  or in compact form
  \[
  Q(a, b) = \frac{1 + R(a, b) - R(b, a)}{2}
  \]
2.2 Transitivity of crisp and fuzzy relations

Transitivity of a crisp relation $R$:

$$(R(a, b) = 1 \land R(b, c) = 1) \Rightarrow R(a, c) = 1$$
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Transitivity of a crisp relation $R$:

\[(R(a, b) = 1 \land R(b, c) = 1) \Rightarrow R(a, c) = 1\]

$T$-transitivity of a fuzzy relation $R$: t-norm $T$

\[T(R(a, b), R(b, c)) \leq R(a, c)\]
2.2 Triangular norms

- **Triangular norm** (t-norm): $T : [0, 1]^2 \rightarrow [0, 1]$ such that
  - increasing, neutral element $1$ (and absorbing element $0$)
  - commutative and associative
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- Basic t-norms (algebraic importance):

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</tr>
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- **Ordering of basic t-norms:** $T_L \leq T_P \leq T_M$
2.2 Triangular norms

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- Ordering of basic t-norms: $T_L \leq T_P \leq T_M$

- **Min-transitivity**: equivalent to transitivity of cut relations $R_\alpha$
2.3 Stochastic transitivity of reciprocal relations

Increasing $g : [1/2, 1]^2 \rightarrow [0, 1]$ such that $g(1/2, 1/2) \leq 1/2$. 
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- Increasing $g : [1/2, 1]^2 \rightarrow [0, 1]$ such that $g(1/2, 1/2) \leq 1/2$.

- A reciprocal relation $Q$ on $A$ is called $g$-stochastic transitive if

$(Q(a, b) \geq 1/2 \land Q(b, c) \geq 1/2) \Rightarrow g(Q(a, b), Q(b, c)) \leq Q(a, c)$

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<th>Type of Stochastic Transitivity</th>
<th>Corresponding Value</th>
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<td>min</td>
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2.3 Stochastic transitivity of reciprocal relations

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- A reciprocal relation $Q$ on $A$ is called $g$-stochastic transitive if

\[
(Q(a, b) \geq 1/2 \land Q(b, c) \geq 1/2) \Rightarrow g(Q(a, b), Q(b, c)) \leq Q(a, c)
\]

- **Strong stochastic transitivity**
  - $\max$

- **Moderate stochastic transitivity**
  - $\min$

- **Weak stochastic transitivity**
  - $1/2$

- Moderate stochastic transitivity: equivalent to transitivity of cut relations $Q_\alpha$, $\alpha \geq 1/2$
Increasing $h : [1/2, 1]^2 \rightarrow [0, 1]$ such that $h(1/2, 1/2) \leq 1/2$ and $h(1/2, 1) = h(1, 1/2) = 1$. 
2.3 Isostochastic transitivity of reciprocal relations

Increasing \( h : [1/2, 1]^2 \rightarrow [0, 1] \) such that \( h(1/2, 1/2) \leq 1/2 \) and \( h(1/2, 1) = h(1, 1/2) = 1 \).

A reciprocal relation \( Q \) on \( A \) is called \( h \)-isostochastic transitive if

\[
(Q(a, b) \geq 1/2 \land Q(b, c) \geq 1/2) \Rightarrow h(Q(a, b), Q(b, c)) = Q(a, c)
\]
2.4 $FG$-transitivity of reciprocal relations

Two $[1/2, 1]^2 \rightarrow [0, 1]$ mappings $F$ and $G$ such that

- $F(1/2, 1/2) \leq 1/2 \leq G(1/2, 1/2)$ and
- $G(1/2, 1) = G(1, 1/2) = G(1, 1) = 1$
- $F \leq G$
2.4 $FG$-transitivity of reciprocal relations

- Two $[1/2, 1]^2 \rightarrow [0, 1]$ mappings $F$ and $G$ such that
  - $F(1/2, 1/2) \leq 1/2 \leq G(1/2, 1/2)$ and
  - $G(1/2, 1) = G(1, 1/2) = G(1, 1) = 1$
  - $F \leq G$

- A reciprocal relation $Q$ on $A$ is called $FG$-transitive if

\[
(Q(a, b) \geq 1/2 \land Q(b, c) \geq 1/2) \\
\Downarrow \\
F(Q(a, b), Q(b, c)) \leq Q(a, c) \leq G(Q(a, b), Q(b, c))
\]
### 2.4 $FG$-transitivity is a general framework

**Stochastic transitivity:**

<table>
<thead>
<tr>
<th>Type</th>
<th>Condition</th>
<th>Value</th>
</tr>
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<tbody>
<tr>
<td>$g$-stochastic transitivity</td>
<td>$F = g$</td>
<td>$G = 1$</td>
</tr>
<tr>
<td>$h$-isostochastic transitivity</td>
<td>$F = h$</td>
<td>$G = h$</td>
</tr>
</tbody>
</table>
2.4 \textit{FG}-transitivity is a general framework

- Stochastic transitivity:

\begin{align*}
\text{\textit{g}-stochastic transitivity} & : F = g & G = 1 \\
\text{\textit{h}-isostochastic transitivity} & : F = h & G = h
\end{align*}

- \textit{T}-transitivity, with \textit{T} a left-continuous t-norm = \textit{FG}-transitivity w.r.t.

\[
F(x, y) = \max(T(x, y), 1 - I_T(x, 1 - y), 1 - I_T(y, 1 - x)) \\
G(x, y) = 1 - T(1 - x, 1 - y)
\]

with \( I_T \) the residual implicator of \textit{T} defined by

\[
I_T(x, y) = \sup \{ z \in [0, 1] \mid T(x, z) \leq y \} \]
3.1 Cycle-transitivity

Framework for studying the transitivity of reciprocal relations
3.1 Cycle-transitivity

Framework for studying the transitivity of reciprocal relations

Unorthodox evaluation:

- triangles are visited in a cyclic manner
- while ordering the weights encountered
- and imposing an upper bound on sum minus 1
3.1 Cycle-transitivity

- Framework for studying the transitivity of reciprocal relations
- Unorthodox evaluation:
  - triangles are visited in a cyclic manner
  - while ordering the weights encountered
  - and imposing an upper bound on sum minus 1
- Harbours various types of fuzzy and stochastic transitivity
3.1 Cycle-transitivity

Reciprocal relation $Q$:

<table>
<thead>
<tr>
<th>$\alpha_{abc}$</th>
<th>$\min{Q(a, b), Q(b, c), Q(c, a)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{abc}$</td>
<td>$\text{median}{Q(a, b), Q(b, c), Q(c, a)}$</td>
</tr>
<tr>
<td>$\gamma_{abc}$</td>
<td>$\max{Q(a, b), Q(b, c), Q(c, a)}$</td>
</tr>
</tbody>
</table>

![Diagram of cycle-transitivity with red arrows and labels $\alpha$, $\beta$, $\gamma$.]
A reciprocal relation $Q$ on $A$ is called **cycle-transitive** w.r.t. an upper bound function $U$ if for any $a, b, c \in A$

$$L(\alpha_{abc}, \beta_{abc}, \gamma_{abc}) \leq \alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 \leq U(\alpha_{abc}, \beta_{abc}, \gamma_{abc})$$
3.1 Cycle-transitivity

A reciprocal relation $Q$ on $A$ is called cycle-transitive w.r.t. an upper bound function $U$ if for any $a, b, c \in A$

$$L(\alpha_{abc}, \beta_{abc}, \gamma_{abc}) \leq \alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 \leq U(\alpha_{abc}, \beta_{abc}, \gamma_{abc})$$

Dual lower bound function: function $L : \Delta \rightarrow \mathbb{R}$ defined by

$$L(\alpha, \beta, \gamma) = 1 - U(1 - \gamma, 1 - \beta, 1 - \alpha)$$
3.1 Cycle-transitivity

A reciprocal relation $Q$ on $A$ is called cycle-transitive w.r.t. an upper bound function $U$ if for any $a, b, c \in A$

$$L(\alpha_{abc}, \beta_{abc}, \gamma_{abc}) \leq \alpha_{abc} + \beta_{abc} + \gamma_{abc} - 1 \leq U(\alpha_{abc}, \beta_{abc}, \gamma_{abc})$$

Dual lower bound function: function $L : \Delta \rightarrow \mathbb{R}$ defined by

$$L(\alpha, \beta, \gamma) = 1 - U(1 - \gamma, 1 - \beta, 1 - \alpha)$$

A function $U : \Delta = \{(x, y, z) \in [0, 1]^3 \mid x \leq y \leq z\} \rightarrow \mathbb{R}$ is called an upper bound function if it satisfies:

- $U(0, 0, 1) \geq 0$ and $U(0, 1, 1) \geq 1$
- for any $(\alpha, \beta, \gamma) \in \Delta$:
  $$U(\alpha, \beta, \gamma) \geq 1 - U(1 - \gamma, 1 - \beta, 1 - \alpha)$$
3.2 Stochastic transitivity

- **Commutative** $g$ such that $g(1/2, x) \leq x$.

- **Theorem**: $g$-stochastic transitivity = cycle-transitivity w.r.t.

$$U_g(\alpha, \beta, \gamma) = \begin{cases} 
\beta + \gamma - g(\beta, \gamma) & , \text{if } \beta \geq 1/2 \land \alpha < 1/2 \\
1/2 & , \text{if } \alpha \geq 1/2 \\
2 & , \text{if } \beta < 1/2 
\end{cases}$$

<table>
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<tr>
<th>type</th>
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<th>equivalent</th>
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<tr>
<td>strong</td>
<td>$\beta$</td>
<td>$\beta$, if $\beta \geq 1/2$</td>
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<td>moderate</td>
<td>$\gamma$</td>
<td></td>
</tr>
<tr>
<td>weak</td>
<td>$\beta + \gamma - 1/2$</td>
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3.3 Partial stochastic transitivity

A reciprocal relation $Q$ on $A$ is called partial stochastic transitive if

$$(Q(a, b) > 1/2 \land Q(b, c) > 1/2) \Rightarrow \min(Q(a, b), Q(b, c)) \leq Q(a, c)$$
3.3 Partial stochastic transitivity

A reciprocal relation $Q$ on $A$ is called **partial stochastic transitive** if

$$(Q(a, b) > 1/2 \land Q(b, c) > 1/2) \Rightarrow \min(Q(a, b), Q(b, c)) \leq Q(a, c)$$

Slightly weaker than moderate stochastic transitivity
3.3 Partial stochastic transitivity

A reciprocal relation $Q$ on $A$ is called **partial stochastic transitive** if

$$(Q(a, b) > 1/2 \land Q(b, c) > 1/2) \Rightarrow \min(Q(a, b), Q(b, c)) \leq Q(a, c)$$

Slightly weaker than moderate stochastic transitivity

Partial stochastic transitivity = cycle-transitivity w.r.t.

$$U(\alpha, \beta, \gamma) = \gamma$$
3.4 $T$-transitivity of reciprocal relations

1-Lipschitz t-norm $T$:

$$|T(x_1, y_1) - T(x_2, y_2)| \leq |x_1 - x_2| + |y_1 - y_2|$$

Theorem: $T$-transitivity = cycle-transitivity w.r.t. $U_T(\alpha, \beta, \gamma) = \alpha + \beta - T(\alpha, \beta)$

<table>
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<th>t-norm</th>
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<td>$T_M$</td>
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<td>$\beta$</td>
</tr>
<tr>
<td>$T_P$</td>
<td>$\alpha + \beta - \alpha\beta$</td>
<td></td>
</tr>
<tr>
<td>$T_L$</td>
<td>$\min(\alpha + \beta, 1)$</td>
<td>$1$</td>
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3.4 Copulas

**Copula:** $C : [0, 1]^2 \rightarrow [0, 1]$ such that
- neutral element 1, absorbing element 0
- moderate growth:

$$((x_1 \leq x_2 \land y_1 \leq y_2) \Rightarrow C(x_1, y_1) + C(x_2, y_2) \geq C(x_1, y_2) + C(x_2, y_1))$$
3.4 Copulas

- **Copula**: $C : [0, 1]^2 \rightarrow [0, 1]$ such that
  - neutral element 1, absorbing element 0
  - moderate growth:

\[
\left( x_1 \leq x_2 \land y_1 \leq y_2 \right) \Rightarrow C(x_1, y_1) + C(x_2, y_2) \geq C(x_1, y_2) + C(x_2, y_1)
\]

- Basic t-norms are copulas and $T_L \leq C \leq T_M$

- Relationship between t-norms and copulas:

  copula + associativity $\Rightarrow$ t-norm

  t-norm + 1-Lipschitz $\Rightarrow$ copula
3.4 The Frank t-norm/copula family

Frank family \((T_s^F)_{s \in [0, \infty]}\): for \(s \in ]0, 1[ \cup ]1, \infty[\)

\[
T_s^F(x, y) = \log_s \left( 1 + \frac{(s^x - 1)(s^y - 1)}{s - 1} \right)
\]
3.4 The Frank $t$-norm/copula family

Frank family $(T^F_s)_{s \in [0, \infty]}$: for $s \in ]0, 1[ \cup ]1, \infty[$

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Limit cases:

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B. De Baets
3.4 The Frank t-norm/copula family

Frank family \((T_s^F)_{s \in [0, \infty]}\): for \(s \in ]0, 1[ \cup ]1, \infty[\)

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Prototypical solutions of the functional equation of Frank:

\[ x + y - T(x, y) = 1 - T(1 - x, 1 - y) \]
3.4 The Frank t-norm/copula family

Frank family \( (T^F_s)_{s \in [0, \infty]}: \) for \( s \in ]0, 1[ \cup ]1, \infty[ \)

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Prototypical solutions of the functional equation of Frank:

\[
x + y - T(x, y) = 1 - T(1 - x, 1 - y)
\]

\( T^F_s \)-transitivity = cycle-transitivity w.r.t.

\[
U_s(\alpha, \beta, \gamma) = 1 - T^F_s(1 - \alpha, 1 - \beta)
\]

B. De Baets
3.5 Self-dual upper bound functions

Self-dual upper bound function \((U = L)\):

\[ U(\alpha, \beta, \gamma) = 1 - U(1 - \gamma, 1 - \beta, 1 - \alpha) \]
3.5 Self-dual upper bound functions

- Self-dual upper bound function \((U = L)\):

\[ U(\alpha, \beta, \gamma) = 1 - U(1 - \gamma, 1 - \beta, 1 - \alpha) \]

- Simplest example: \(U_M = \text{median} \) (\(T_M\)-transitivity)
3.5 Self-dual upper bound functions

Self-dual upper bound function \((U = L)\):

\[
U(\alpha, \beta, \gamma) = 1 - U(1 - \gamma, 1 - \beta, 1 - \alpha)
\]

Simplest example: \(U_M = \text{median} \quad (T_M\text{-transitivity})\)

Other interesting example:

\[
U_E(\alpha, \beta, \gamma) = \alpha \beta + \alpha \gamma + \beta \gamma - 2\alpha \beta \gamma
\]
3.5 Isostochastic transitivity

Commutative $h : [1/2, 1]^2 \rightarrow [1/2, 1]$ with neutral element $1/2$.

Theorem: $h$-isostochastic transitivity = cycle-transitivity w.r.t. the self-dual upper bound function

$$U^i_h(\alpha, \beta, \gamma) = \begin{cases} 
\beta + \gamma - h(\beta, \gamma) & \text{, if } \beta \geq 1/2 \\
\alpha + \beta - 1 + h(1 - \beta, 1 - \alpha) & \text{, if } \beta < 1/2
\end{cases}$$
3.5 Multiplicative transitivity

A reciprocal relation $Q$ on $A$ is called **multiplicatively transitive** if

$$\frac{Q(a, c)}{Q(c, a)} = \frac{Q(a, b)}{Q(b, a)} \cdot \frac{Q(b, c)}{Q(c, b)}$$
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\]

Multiplicative transitivity = $h$-isostochastic transitivity w.r.t.

\[
h(x, y) = \frac{xy}{xy + (1 - x)(1 - y)}
\]
3.5 Multiplicative transitivity

A reciprocal relation $Q$ on $A$ is called multiplicatively transitive if

$$
\frac{Q(a, c)}{Q(c, a)} = \frac{Q(a, b)}{Q(b, a)} \cdot \frac{Q(b, c)}{Q(c, b)}
$$

Multiplicative transitivity = $h$-isostochastic transitivity w.r.t.

$$
h(x, y) = \frac{xy}{xy + (1 - x)(1 - y)}
$$

Multiplicative transitivity = cycle-transitivity w.r.t.

$$
U_E(\alpha, \beta, \gamma) = \alpha \beta + \alpha \gamma + \beta \gamma - 2\alpha \beta \gamma
$$
3.6. $FG$-transitivity versus cycle-transitivity

From $FG$-transitivity to cycle-transitivity:

- **commutative** $[1/2, 1]^2 \rightarrow [0, 1]$ functions $F$ and $G$ such that $F(1/2, x) \leq x \leq G(1/2, x)$

- transformation possible, yet ugly
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From cycle-transitivity to \textit{FG}-transitivity:

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- transformation impossible in essential cases
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From cycle-transitivity to $FG$-transitivity:

- transformation possible in esoteric cases, and difficult
- transformation impossible in essential cases

Counterexample: **dice-transitivity** = cycle-transitivity w.r.t.

\[ U_D(\alpha, \beta, \gamma) = \beta + \gamma - \beta\gamma \]

- between $T_L$-transitivity and $T_P$-transitivity
- between $T_L$-transitivity and moderate stochastic transitivity
4.1 Dice-transitivity of winning probabilities

Random vector $(X, Y)$:

\[ Q(X, Y) = \text{Prob}\{X > Y\} + \frac{1}{2} \text{Prob}\{X = Y\} \]

leads to reciprocity: \[ Q(X, Y) + Q(Y, X) = 1 \]

is in general based on the joint distribution
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leads to reciprocity: \(Q(X, Y) + Q(Y, X) = 1\)

is in general based on the joint distribution

Theorem: The reciprocal relation generated by pairwise independent random variables is dice-transitive, i.e. cycle-transitive w.r.t.

\[
U_D(\alpha, \beta, \gamma) = \beta + \gamma - \beta \gamma
\]
4.1 Dice-transitivity of winning probabilities

Dice-transitivity can be written as:

\[ \beta \gamma \leq 1 - \alpha \]
4.1 Dice-transitivity of winning probabilities

Dice-transitivity can be written as:

\[ \beta \gamma \leq 1 - \alpha \]
4.2 One- and two-parameter families

- Marginal distributions belonging to a same parametric family
  - One-parameter: exponential, geometric, power-law (subfamilies of Beta and Pareto families), Gumbel
    - multiplicative transitivity
  - Two-parameter: normal
    - moderate stochastic transitivity
4.2 One-parameter families

Families with parameter $\lambda \in \mathbb{R}$: $h$-isostochastic transitivity

<table>
<thead>
<tr>
<th>Name</th>
<th>Density function</th>
<th>Parameter $a &gt; 0$</th>
<th>$x \in [\lambda, \lambda + a]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform</td>
<td>$1/a$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Laplace</td>
<td>$e^{-</td>
<td>x-\lambda</td>
<td>/\mu/(2\mu)}$</td>
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</tr>
</thead>
<tbody>
<tr>
<td>Uniform</td>
<td>$1 - 1/2 \left( \max(\sqrt{2(1-x)} + \sqrt{2(1-y)} - 1, 0) \right)^2$</td>
</tr>
<tr>
<td>Laplace</td>
<td>$1 - f^{-1}(f(1-x) + f(1-y))$</td>
</tr>
<tr>
<td></td>
<td>with $f^{-1}(x) = 1/2 (1 + x/2) e^{-x}$</td>
</tr>
<tr>
<td>Normal</td>
<td>$\Phi(\Phi^{-1}(x) + \Phi^{-1}(y))$</td>
</tr>
<tr>
<td></td>
<td>with $\Phi$ the c.d.f. of standard normal distribution</td>
</tr>
</tbody>
</table>
4.3 Sklar’s theorem

Sklar’s theorem: for a random vector \((X_1, X_2, \ldots, X_n)\) there exist copulas \(C_{ij}\) s.t.

\[
F_{X_i, X_j}(x, y) = C_{ij}(F_{X_i}(x), F_{X_j}(y))
\]
4.3 Sklar’s theorem

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Captures dependence structure irrespective of the marginals

Probabilistic interpretation:

| \(T_M\) | co-monotonicity |
| \(T_P\) | independence |
| \(T_L\) | counter-monotonicity |
4.3 Example

\[ Q^P(X, Y) = \frac{7}{16} \]
\[ Q^M(X, Y) = \frac{3}{8} \]
\[ Q^L(X, Y) = \frac{1}{2} \]
The compatibility problem:

- not all combinations of copulas are possible
- all $C_{ij} = C$ is possible for $C \in \{T_M, T_P\}$
- $C_{12} = C_{13} = C_{23} = T_L$ is impossible
4.4 Dependence and the compatibility problem

The compatibility problem:

- not all combinations of copulas are possible
- all \( C_{ij} = C \) is possible for \( C \in \{ T_M, T_P \} \)
- \( C_{12} = C_{13} = C_{23} = T_L \) is impossible

Artificial coupling:

- winning probabilities require only bivariate coupling
- copula = comparison strategy
- does not (necessarily) reflect the real dependence
4.4 Coupling by the same copula: cycle-transitivity

Stable commutative copula:

\[ x + y - C(x, y) = 1 - C(1 - x, 1 - y) \]
4.4 Coupling by the same copula: cycle-transitivity

- **Stable commutative copula:**
  \[ x + y - C(x, y) = 1 - C(1 - x, 1 - y) \]

- **Theorem:** for a stable commutative copula \( C \), the reciprocal relation \( Q^C \) is cycle-transitive w.r.t.
  \[ U^C(\alpha, \beta, \gamma) = \gamma + C(\beta, 1 - \gamma) \]
4.4 Coupling by a Frank copula

Frank t-norms/copulas are stable commutative copulas:

\[ U^s(\alpha, \beta, \gamma) = \beta + \gamma - T^F_{1/s}(\beta, \gamma) \]

<table>
<thead>
<tr>
<th>copula</th>
<th>upper bound f.</th>
<th>equivalent</th>
<th>known as</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_M )</td>
<td>( \min(\beta + \gamma, 1) )</td>
<td>1</td>
<td>( T_L )-transitivity</td>
</tr>
<tr>
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<td>partial ST</td>
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5.1 Stochastic dominance

Purpose of stochastic dominance:

- to define a (partial) order relation on a set of real-valued random variables (RV)
- endowed with the semantics of a weak preference relation:

  RV taking higher values are preferred
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Purpose of stochastic dominance:

- to define a (partial) order relation on a set of real-valued random variables (RV)
- endowed with the semantics of a weak preference relation:

  RV taking higher values are preferred

General principle:

- pairwise comparison of RV
- pointwise comparison of performance functions
- constructed from the distribution function
5.1 Comparison functions

- The cumulative distribution function (CDF) $F_X$:
  \[ F_X(x) = \text{Prob}\{X \leq x\} \]

- The area below the CDF $F_X$:
  \[ G_X(x) = \int_{-\infty}^{x} F_X(t) \, dt \]
### 5.1 1st and 2nd degree stochastic dominance (SD)

#### Weak dominance relation:

| $X \succeq_{FSD} Y$ | $\overset{\text{def}}{\iff} | F_X \leq F_Y |
|----------------------|--------------------------------|
| $\iff | E[u(X)] \geq E[u(Y)] |
| for any non-decreasing function $u$ |

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**Strict dominance relation:**

$X \succ Y \iff X \succeq Y$ and $Y \not\succeq X$
5.1 Graphical illustration of FSD
5.2 Application areas

- Decision making under uncertainty

- Risk averse preference models in economics and finance:
  - e.g. in portfolio optimisation

- Social statistics:
  - e.g. in the comparison of welfare and poverty indicators

- Machine learning and multi-criteria decision making:
  - e.g. in ranking (= ordered sorting) algorithms
5.3 Discussion

SD induces a **crisp partial order relation** on a set of RV:

- **crisp**: no tolerance for small deviations, **no grading**
- **partial**: usually **sparse** graphs
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- SD is theoretically attractive, but **computationally difficult**

- SD uses **marginal distributions** only:
  - does not take into account **dependence** between RV

- SSD accumulates area from $-\infty$ onwards
  - introduces an **absolute reference point**
5.3 Main objective: graded variants of SD

Pairwise construction of a transitive $[0, 1]$-valued relation on a set of RV which:

- avoids the pointwise comparison of performance functions
- allows to incorporate dependence between the RV
- avoids specific reference points
- allows to induce a strict order relation on the set of RV
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- Choose a stable commutative copula \( C \) as comparison strategy and compute:

\[
Q^C(X, Y) = \text{Prob}\{X > Y\} + \frac{1}{2} \text{Prob}\{X = Y\}
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- The reciprocal relation \(Q^C\) is cycle-transitive w.r.t.

\[
U^C(\alpha, \beta, \gamma) = \gamma + C(\beta, 1 - \gamma)
\]
5.4 Example: co-monotone comparison

The case $T_M$: continuous RV

$$Q^M(X, Y) = \int_{x:F_X(x)<F_Y(x)} f_X(x) \, dx + \frac{1}{2} \int_{x:F_X(x)=F_Y(x)} f_X(x) \, dx$$

$Q^M(X, Y) = 1$ iff $F_X < F_Y$ where $f_X \neq 0$: more restrictive than $\succ_FSD$
5.4 Graphical illustration

\[ Q^M(X, Y) = t_1 + t_3 + \frac{1}{2} t_2 \]
6.1 Statistical preference and stochastic dominance

**Statistical preference:** \( X \succsim Y \) if \( Q^P(X, Y) \geq 1/2 \)
6.1 Statistical preference and stochastic dominance

- **Statistical preference:** $X \succeq Y$ if $Q^p(X, Y) \geq 1/2$

- **Theorem:** FSD implies statistical preference
6.1 Statistical preference and stochastic dominance

- **Statistical preference**: \( X \succeq Y \) if \( Q^P(X, Y) \geq 1/2 \)

- **Theorem**: FSD implies statistical preference
6.2 Exploiting cycle-transitivity: $T_P$

- The relation $\succ^3_P$

The relation $X \succ^3_P Y \iff Q^P(X, Y) > \frac{\sqrt{5} - 1}{2}$

is an asymmetric relation without cycles of length 3

- The golden section

The golden section $\frac{\sqrt{5} - 1}{2} : \frac{22}{36} < \frac{\sqrt{5} - 1}{2} < \frac{23}{36}$

B. De Baets
6.2 Exploiting cycle-transitivity: $T_P$

The relation $>^k_P$

$$X >^k_P Y \iff Q^P(X, Y) > 1 - \frac{1}{4 \cos^2\left(\frac{\pi}{(k + 2)}\right)}$$

is an asymmetric relation without cycles of length $k$. 
The relation $\succ^k_P$:

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The relation $\succ^\infty_P$:

$$X \succ^\infty_P Y \iff Q^P(X, Y) \geq \frac{3}{4}$$

is an asymmetric acyclic relation.
6.2 Exploiting cycle-transitivity: $T_P$

- The relation $\succ^k_P$
  \[ X \succ^k_P Y \iff Q^P(X, Y) > 1 - \frac{1}{4 \cos^2\left(\frac{\pi}{k + 2}\right)} \]

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- The relation $\succ^\infty_P$
  \[ X \succ^\infty_P Y \iff Q^P(X, Y) \geq \frac{3}{4} \]

  is an asymmetric acyclic relation

- The transitive closure $\succ_P$ of $\succ^\infty_P$ is a strict order relation
The relation $\overset{k}{\succ}_M$ is an asymmetric relation without cycles of length $k$. 

\[
X \overset{k}{\succ}_M Y \iff Q^M(X, Y) > \frac{k - 1}{k}
\]
6.2 Exploiting cycle-transitivity: $T_M$ and $T_L$

The relation $\succ^k_M$

\[ X \succ^k_M Y \iff Q^M(X, Y) > \frac{k - 1}{k} \]

is an asymmetric relation without cycles of length $k$.

The relation $\succ_M$

\[ X \succ_M Y \iff Q^M(X, Y) = 1 \]

is a strict order relation.
6.2 Exploiting cycle-transitivity: $T_M$ and $T_L$

- The relation $\succ^k_M$

\[ X \succ^k_M Y \iff Q^M(X, Y) > \frac{k - 1}{k} \]

is an asymmetric relation **without cycles of length** $k$

- The relation $\succ_M$

\[ X \succ_M Y \iff Q^M(X, Y) = 1 \]

is a **strict order relation**

- The relation $\succ_L$

\[ X \succ_L Y \iff Q^L(X, Y) > \frac{1}{2} \]

is a **strict order relation**
### Cutting levels:

<table>
<thead>
<tr>
<th>copula</th>
<th>$s$</th>
<th>level $\alpha_s$</th>
</tr>
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<tbody>
<tr>
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6.3 The Frank copula family

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The Frank copula family:

$$\alpha_s = 1 - \log_s \left( \frac{1 + \sqrt{s}}{2} \right)$$

with

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6.3 The Frank copula family

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The Frank copula family:

\[
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\]

with

\[
\alpha_s + \alpha_{1/s} = \frac{3}{2}
\]

None of the strict order relations \( >_s \) generalizes strict FSD
6.3 A picture says more than . . .
Reciprocal relation: \[ Q^M(X, Y) = \frac{1}{n} \sum_{k=1}^{n} \delta^M_k \]

with

\[ \delta^M_k = \begin{cases} 
1 & \text{, if } x(k) > y(k) \\ 
1/2 & \text{, if } x(k) = y(k) \\ 
0 & \text{, if } x(k) < y(k) 
\end{cases} \]
6.4 Co-monotone comparison revisited

Reciprocal relation:

\[ Q^M(X, Y) = \frac{1}{n} \sum_{k=1}^{n} \delta_k^M \]

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\end{cases} \]

Parametrized version: \( p \in \mathbb{R}^+ \)

\[ Q^M_p(X, Y) = \frac{\sum_{k=1}^{n} (x(k) - y(k))^p}{\sum_{k=1}^{n} |x(k) - y(k)|^p} \]
Limit case: $Q_0^M = Q^M$
6.4 Co-monotone comparison revisited

Limit case: $Q_0^M = Q^M$

The case of continuous RV and $p = 1$:

$$Q_1^M(X, Y) = \frac{\int (F_Y(x) - F_X(x))_+ \, dx}{\int |F_Y(x) - F_X(x)| \, dx}$$
6.4 Co-monotone comparison revisited

- Limit case: $Q_0^M = Q^M$

- The case of continuous RV and $p = 1$:
  
  $Q_1^M(X, Y) = \frac{\int (F_Y(x) - F_X(x))_+ \, dx}{\int |F_Y(x) - F_X(x)| \, dx}$

- $Q_1^M(X, Y) = 1$ iff $X \succ_{\text{FSD}} Y$
6.4 Graphical illustration

![Graphical illustration](image-url)
6.4 Transitivity

- **Theorem**: the probabilistic relation $Q_1^M$ is moderately stochastic transitive

- The strict order relation at $1/2$:

\[ Q_1^M(X, Y) > \frac{1}{2} \iff E[X] > E[Y] \]

- Any weak ($> 1/2$) or strict ($\geq 1/2$) cutting level $\alpha$ yields a strict order relation:

  - with increasing $\alpha$ the graph become more and more sparse (Hasse tree)
7. Conclusion

- **Cycle-transitivity** = general framework for studying the transitivity of reciprocal relations

- **Unorthodox evaluation**: triangles are visited in a cyclic manner, while ordering the weights encountered, and imposing an upper bound on ‘sum minus 1’

- Allows the description of the transitivity of winning probabilities among (artificially coupled) random variables

- Starting point for the development of alternative notions of stochastic dominance
7. Conclusion

Question 1:
Does there exist a general framework for studying transitivity?
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Answer: YES
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Are cycles really incompatible with any notion of transitivity?
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Does there exist a general framework for studying transitivity?
Answer: YES

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Question 3:
Can cycles be resolved?
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Can cycles be resolved?

Answer: YES
Thank you for your attention!

{Bernard.DeBaets,Hans.DeMeyer}@UGent.be